ULB/229/CQ/97/4

Three-body Generalizations of the Sutherland Problem

C. Quesne*

Physique Nucléaire Théorique et Physique Mathématique,

Université Libre de Bruxelles, Campus de la Plaine CP229,

Boulevard du Triomphe, B-1050 Brussels, Belgium

Abstract

The three-particle Hamiltonian obtained by replacing the two-body trigonometric potential of the Sutherland problem by a three-body one of a similar form is shown to be exactly solvable. When written in appropriate variables, its eigenfunctions can be expressed in terms of Jack symmetric polynomials. The exact solvability of the problem is explained by a hidden $sl(3, \mathbf{R})$ symmetry. A generalized Sutherland three-particle problem including both two- and three-body trigonometric potentials and internal degrees of freedom is then considered. It is analyzed in terms of three first-order noncommuting differential-difference operators, which are constructed by combining SUSYQM supercharges with the elements of the dihedral group D_6. Three alternative commuting operators are also introduced.

1 Introduction

In 1974, Calogero and Marchioro [Cal74], on one hand, and Wolfes [Wol74], on the other hand, extended the Calogero problem [Cal69] for three particles on a line interacting via inverse-square two-body potentials (and harmonic forces in the case of bound states) to a problem where there is an additional three-body potential of a similar form. Later on, it was pointed out by Olshanetsky and Perelomov [Ols83] that the Calogero-Marchioro-Wolfes (CMW) problem is related to the root system of the exceptional Lie algebra G_2, and to the Weyl group of the latter, namely the dihedral

*Directeur de recherches FNRS; E-mail: cquesne@ulb.ac.be

group D_6. More recently, the Brink *et al* [Bri92] and Polychronakos [Pol92] exchange operator formalism was extended to the CMW problem to deal with particles with internal degrees of freedom, thereby leading to a D_6-extended Heisenberg algebra [Que95].

In this communication, we present some results for similar generalizations of the Sutherland problem [Sut71], wherein trigonometric potentials are considered instead of inverse-square ones [Que96, Que97]. The Hamiltonian considered here is

$$\begin{aligned} H &= -\sum_{i=1}^{3} \partial_i^2 + ga^2 \sum_{\substack{i,j=1 \\ i\neq j}}^{3} \csc^2\left(a(x_i - x_j)\right) \\ &\quad + 3fa^2 \sum_{\substack{i,j,k=1 \\ i\neq j\neq k\neq i}}^{3} \csc^2\left(a(x_i + x_j - 2x_k)\right), \end{aligned} \tag{1}$$

where x_i, $i = 1, 2, 3$, $0 \le x_i \le \pi/a$, denote the particle coordinates, $\partial_i \equiv \partial/\partial x_i$, and g, f are assumed not to vanish simultaneously and to be such that $g > -1/4$, $f > -1/4$. In the case where $g \neq 0$ and $f = 0$, Hamiltonian (1) reduces to the Sutherland Hamiltonian [Sut71], while for $a \to 0$, it goes over into the CMW Hamiltonian [Cal74, Wol74].

Hamiltonian (1) is invariant under translations of the centre-of-mass, whose coordinate will be denoted by $R = (x_1 + x_2 + x_3)/3$. In other words, H commutes with the total momentum $P = -i\sum_{i=1}^{3} \partial_i$, which may be simultaneously diagonalized. It proves convenient to use two different systems of relative coordinates, namely $x_{ij} \equiv x_i - x_j$, $i \neq j$, and $y_{ij} \equiv x_i + x_j - 2x_k$, $i \neq j \neq k \neq i$, where in the latter, we suppressed index k as it is entirely determined by i and j.

Since the potentials are singular and crossing is therefore not allowed, in the case of distinguishable particles the wave functions in different sectors of configuration space are disconnected, while for indistinguishable particles, they are related by a symmetry requirement.

For distinguishable particles in a given sector of configuration space, the unnormalized ground-state wave function of Hamiltonian (1) is given by

$$\psi_0(\boldsymbol{x}) = \prod_{\substack{i,j=1 \\ i\neq j}}^{3} \left|\sin(ax_{ij})\right|^{\kappa} \left|\sin(ay_{ij})\right|^{\lambda}, \tag{2}$$

where $\kappa \equiv \left(1 + \sqrt{1+4g}\right)/2$ or 0, and $\lambda \equiv \left(1 + \sqrt{1+4f}\right)/2$ or 0, according to whether $g \neq 0$ or $g = 0$, and $f \neq 0$ or $f = 0$, respectively (or, equivalently, $g = \kappa(\kappa - 1)$, $f = \lambda(\lambda - 1)$). The corresponding eigenvalues of H and P are $E_0 = 8a^2(\kappa^2 + 3\kappa\lambda + 3\lambda^2)$, and $p_0 = 0$ [Que96].

In Sec. 2, we will prove that Hamiltonian (1) with pure three-body interactions, i.e., for $g = 0$, is exactly solvable, and we will derive its

energy spectrum and eigenfunctions. In Sec. 3, we will propose an extension of (1) for a system of three particles with internal degrees of freedom, and introduce the corresponding exchange operator formalism.

2 Exact Solvability of the Pure Three-Body Problem

Let us assume that $g = 0$ (hence $\kappa = 0$), and $f \neq 0$ in Eq. (1). For distinguishable particles in a given sector of configuration space, the simultaneous solutions of the eigenvalue equations $H\psi(\boldsymbol{x}) = E\psi(\boldsymbol{x})$, and $P\psi(\boldsymbol{x}) = p\psi(\boldsymbol{x})$ can be found by setting $\psi(\boldsymbol{x}) = \psi_0(\boldsymbol{x})\varphi(\boldsymbol{x})$. The functions $\varphi(\boldsymbol{x})$ satisfy the equations $h\varphi(\boldsymbol{x}) = \epsilon\varphi(\boldsymbol{x})$, and $P\varphi(\boldsymbol{x}) = p\varphi(\boldsymbol{x})$, where $h \equiv (\psi_0(\boldsymbol{x}))^{-1}(H - E_0)\psi_0(\boldsymbol{x})$, and $\epsilon \equiv E - E_0$. In terms of the new variables $z_i \equiv \exp\left(\frac{2}{3}ia(x_i - 2x_j + 4x_k)\right)$, where $(ijk) = (123)$, the gauge-transformed Hamiltonian h becomes

$$h = 12a^2 \left(\sum_i \left(z_i \partial_{z_i}\right)^2 + \lambda \sum_{\substack{i,j \\ i \neq j}} \frac{z_i + z_j}{z_i - z_j} z_i \partial_{z_i} \right) - \frac{8}{3} a^2 \left(\sum_i z_i \partial_{z_i} \right)^2, \quad (3)$$

while $P = 2a \sum_i z_i \partial_{z_i}$.

It can be easily proved [Que97] that the eigenfunctions and eigenvalues of h and P are given by

$$\varphi_{\{k\}}(\boldsymbol{x}) = \exp(6iaqR) J_{\{\mu\}}\left(\boldsymbol{z}; \lambda^{-1}\right), \quad (4)$$

and

$$\begin{aligned} \epsilon_{\{k\}} &= 4a^2 \left[3 \sum_i k_i^2 - \tfrac{2}{3} \left(\sum_i k_i \right)^2 - 6\lambda^2 \right], \\ p_{\{k\}} &= 2a \sum_i k_i = 2a \left(\sum_i \mu_i + 3q \right), \end{aligned} \quad (5)$$

where $J_{\{\mu\}}\left(\boldsymbol{z}; \lambda^{-1}\right)$ denotes the Jack (symmetric) polynomial in the variables z_i, $i = 1, 2, 3$, corresponding to the parameter λ^{-1}, and the partition $\{\mu\} = \{\mu_1 \mu_2\}$ into not more than two parts [Sta89]. In Eqs. (4) and (5), $k_1 = q - \lambda$, $k_2 = \mu_2 + q$, $k_3 = \mu_1 + q + \lambda$, and $q \in \mathbb{R}$. In Table 1, the explicit form of $J_{\{\mu\}}\left(\boldsymbol{z}; \lambda^{-1}\right)$ is given for $\mu_1 + \mu_2 \leq 4$.

The eigenfunctions of h can be separated into centre-of-mass and relative functions as follows:

$$\varphi_{\{k\}}(\boldsymbol{x}) = \exp\left[2ia \left(\sum_i k_i \right) R \right] P_{\{\mu\}}\left(\boldsymbol{\zeta}; \lambda^{-1}\right), \quad (6)$$

Table 1: Jack polynomials $J_{\{\mu\}}\left(\boldsymbol{z};\lambda^{-1}\right)$ for $\mu_1+\mu_2 \le 4$.

$\{\mu\}$	$J_{\{\mu\}}\left(\boldsymbol{z};\lambda^{-1}\right)$
$\{0\}$	1
$\{1\}$	$\sum_i z_i$
$\{1^2\}$	$\sum_{i<j} z_i z_j$
$\{2\}$	$\sum_i z_i^2 + \frac{2\lambda}{\lambda+1} \sum_{i<j} z_i z_j$
$\{21\}$	$\sum_{i\ne j} z_i^2 z_j + \frac{6\lambda}{2\lambda+1} z_1 z_2 z_3$
$\{3\}$	$\sum_i z_i^3 + \frac{3\lambda}{\lambda+2} \sum_{i\ne j} z_i^2 z_j + \frac{6\lambda^2}{(\lambda+1)(\lambda+2)} z_1 z_2 z_3$
$\{2^2\}$	$\sum_{i<j} z_i^2 z_j^2 + \frac{2\lambda}{\lambda+1} z_1 z_2 z_3 \sum_i z_i$
$\{31\}$	$\sum_{i\ne j} z_i^3 z_j + \frac{2\lambda}{\lambda+1} \sum_{i<j} z_i^2 z_j^2 + \frac{\lambda(5\lambda+3)}{(\lambda+1)^2} z_1 z_2 z_3 \sum_i z_i$
$\{4\}$	$\sum_i z_i^4 + \frac{4\lambda}{\lambda+3} \sum_{i\ne j} z_i^3 z_j + \frac{6\lambda(\lambda+1)}{(\lambda+2)(\lambda+3)} \sum_{i<j} z_i^2 z_j^2$
	$+ \frac{12\lambda^2}{(\lambda+2)(\lambda+3)} z_1 z_2 z_3 \sum_i z_i$

where $P_{\{\mu\}}\left(\boldsymbol{\zeta};\lambda^{-1}\right)$ is the polynomial in $\zeta_1 \equiv \sum_i v_i$ and $\zeta_2 \equiv \sum_{i<j} v_i v_j$, $v_i \equiv \exp(-2iax_{jk}) = z_i \exp(-2iaR)$ for $(ijk) = (123)$, obtained from the corresponding Jack polynomial $J_{\{\mu\}}\left(\boldsymbol{v};\lambda^{-1}\right)$ by making the change of variables $v_i \to \zeta_1$, ζ_2. It satisfies the eigenvalue equation

$$h^{rel} P_{\{\mu\}}\left(\boldsymbol{\zeta};\lambda^{-1}\right) = \epsilon^{rel}_{\{\mu\}} P_{\{\mu\}}\left(\boldsymbol{\zeta};\lambda^{-1}\right), \tag{7}$$

where

$$\begin{aligned} h^{rel} &= 8a^2 \Big[\left(\zeta_1^2 - 3\zeta_2\right) \partial^2_{\zeta_1} + (\zeta_1\zeta_2 - 9)\partial^2_{\zeta_1\zeta_2} + \left(\zeta_2^2 - 3\zeta_1\right) \partial^2_{\zeta_2} \\ &\quad + (3\lambda+1)\left(\zeta_1 \partial_{\zeta_1} + \zeta_2 \partial_{\zeta_2}\right) \Big], & (8) \\ \epsilon^{rel}_{\{\mu\}} &= 8a^2 \left(\mu_1^2 - \mu_1\mu_2 + \mu_2^2 + 3\lambda\mu_1\right). & (9) \end{aligned}$$

The relative energies are similar to those obtained with pure two-body interactions, i.e. for $g \ne 0$ and $f = 0$. In Table 2, they are listed for $\mu_1 + \mu_2 \le 4$, together with the corresponding eigenfunctions $P_{\{\mu\}}\left(\boldsymbol{\zeta};\lambda^{-1}\right)$. On the results displayed in the Table, it can be checked that $P_{\{\mu\}}\left(\boldsymbol{\zeta};\lambda^{-1}\right)$ belongs to the space $V_{\mu_1}(\boldsymbol{\zeta})$, where $V_n(\boldsymbol{\zeta})$, $n \in \mathbb{N}$, is defined as the space of

polynomials in ζ_1 and ζ_2 that are of degree less than or equal to n (hence, $\dim V_n = (n+1)(n+2)/2$).

Table 2: Eigenvalues $\epsilon^{rel}_{\{\mu\}}$ and eigenfunctions $P_{\{\mu\}}\left(\boldsymbol{\zeta};\lambda^{-1}\right)$ of h^{rel} for $\mu_1 + \mu_2 \leq 4$.

$\{\mu\}$	$\epsilon^{rel}_{\{\mu\}}/(8a^2)$	$P_{\{\mu\}}\left(\boldsymbol{\zeta};\lambda^{-1}\right)$
$\{0\}$	0	1
$\{1\}$	$3\lambda+1$	ζ_1
$\{1^2\}$	$3\lambda+1$	ζ_2
$\{2\}$	$2(3\lambda+2)$	$\zeta_1^2 - \frac{2}{\lambda+1}\zeta_2$
$\{21\}$	$3(2\lambda+1)$	$\zeta_1\zeta_2 - \frac{3}{2\lambda+1}$
$\{3\}$	$9(\lambda+1)$	$\zeta_1^3 - \frac{6}{\lambda+2}\zeta_1\zeta_2 + \frac{6}{(\lambda+1)(\lambda+2)}$
$\{2^2\}$	$2(3\lambda+2)$	$\zeta_2^2 - \frac{2}{\lambda+1}\zeta_1$
$\{31\}$	$9\lambda+7$	$\zeta_1^2\zeta_2 - \frac{2}{\lambda+1}\zeta_2^2 - \frac{3\lambda+1}{(\lambda+1)^2}\zeta_1$
$\{4\}$	$4(3\lambda+4)$	$\zeta_1^4 - \frac{12}{\lambda+3}\zeta_1^2\zeta_2 + \frac{12}{(\lambda+2)(\lambda+3)}(\zeta_2^2 + 2\zeta_1)$

In Ref. [Que97], the degeneracies of the relative energy spectrum (9) were obtained for both distinguishable and indistinguishable (either bosonic or fermionic) particles on the line interval $(0, \pi/a)$, interacting via pure two-body or three-body potential. It was shown that although the results do not depend upon the nature of interactions for distinguishable particles, they do for indistinguishable ones. Such a property is due to the fact that both the configuration space sectors, and the variables the relative wave functions depend upon have different transformation properties under particle permutations for the problems with pure two-body or pure three-body potential.

The exact solvability of H for $g = 0$ and $f \neq 0$, or equivalently of h^{rel}, defined in Eq. (8), can be easily explained by a hidden $sl(3,\mathbb{R})$ symmetry [Que97]. The Hamiltonian h^{rel} can indeed be rewritten as a quadratic combination

$$\begin{aligned} h^{rel} \quad &= \quad 8a^2\big[E_{11}^2 + E_{11}E_{22} + E_{22}^2 - 3E_{12}E_{32} - 3E_{21}E_{31} \\ &\quad - 9E_{31}E_{32} + 3\lambda\left(E_{11} + E_{22}\right)\big] \end{aligned} \tag{10}$$

of the operators

$$\begin{aligned}
E_{11} &= \zeta_1\partial_{\zeta_1}, \qquad E_{22} = \zeta_2\partial_{\zeta_2}, \qquad E_{33} = n - \zeta_1\partial_{\zeta_1} - \zeta_2\partial_{\zeta_2}, \\
E_{21} &= \zeta_2\partial_{\zeta_1}, \qquad E_{12} = \zeta_1\partial_{\zeta_2}, \\
E_{31} &= \partial_{\zeta_1}, \qquad E_{13} = n\zeta_1 - \zeta_1^2\partial_{\zeta_1} - \zeta_1\zeta_2\partial_{\zeta_2}, \\
E_{32} &= \partial_{\zeta_2}, \qquad E_{23} = n\zeta_2 - \zeta_1\zeta_2\partial_{\zeta_1} - \zeta_2^2\partial_{\zeta_2},
\end{aligned} \tag{11}$$

satisfying $gl(3,\mathbb{R})$ commutation relations $[E_{ij}, E_{kl}] = \delta_{kj}E_{il} - \delta_{il}E_{kj}$, together with the constant trace condition $\sum_i E_{ii} = n$ for any real n value. Whenever n is a non-negative integer, the operators E_{ij} preserve the space $V_n(\boldsymbol{\zeta})$. Hence, h^{rel} preserves an infinite flag of spaces, $V_0(\boldsymbol{\zeta}) \subset V_1(\boldsymbol{\zeta}) \subset V_2(\boldsymbol{\zeta}) \subset \ldots$. Its representation matrix is therefore triangular in the basis wherein all spaces $V_n(\boldsymbol{\zeta})$ are naturally defined, so that h^{rel} is exactly solvable. This result is similar to that previously obtained for the pure two-body trigonometric potential [Ruh95].

3 Two- and Three-body Problem with Internal Degrees of Freedom

Let us now assume that both g and f are nonvanishing. From the ground-state wave function (2) of Hamiltonian (1), one can construct SUSYQM supercharge operators $\hat{Q}^+$, $\hat{Q}^- = \left(\hat{Q}^+\right)^\dagger$, whose matrix elements can be expressed in terms of six differential operators $Q_i^\pm = \mp\partial_i - \partial_i \ln\psi_0(\boldsymbol{x})$, $i = 1, 2, 3$ [And84]. The latter are given by [Que96]

$$\begin{aligned}
Q_i^\pm &= \mp\partial_i - \kappa a \sum_{j\neq i} \cot(ax_{ij}) \\
&\quad - \lambda a \left(\sum_{j\neq i} \cot(ay_{ij}) - \sum_{\substack{j,k \\ i\neq j\neq k\neq i}} \cot(ay_{jk}) \right).
\end{aligned} \tag{12}$$

The corresponding supersymmetric Hamiltonian is $\hat{H} = \mathrm{diag}(H^{(0)}, H^{(1)}, H^{(2)}, H^{(3)})$, where $H^{(0)} = H - E_0 = \sum_i Q_i^+ Q_i^-$, $H^{(1)}$ and $H^{(2)}$ contain matrix potentials, while $H^{(3)} = \sum_i Q_i^- Q_i^+$ only differs from $H^{(0)}$ by the replacement in H of $g = \kappa(\kappa-1)$, $f = \lambda(\lambda-1)$ by $g = \kappa(\kappa+1)$, $f = \lambda(\lambda+1)$, respectively.

In the case of the CMW problem, it was shown in Ref. [Que95] that the corresponding operators Q_i^- can be transformed into three commuting differential-difference operators D_i, the so-called Dunkl operators of the mathematical literature [Dun89], by inserting in appropriate places some finite-group elements K_{ij} and $L_{ij} \equiv K_{ij}I_r$. Here K_{ij} are particle permutation operators, while I_r is the inversion operator in relative-coordinate

space. In the centre-of-mass coordinate system, they satisfy the relations

$$\begin{aligned} K_{ij} &= K_{ji} = K_{ij}^{\dagger}, \quad K_{ij}^2 = 1, \quad K_{ij}K_{jk} = K_{jk}K_{ki} = K_{ki}K_{ij}, \\ K_{ij}I_r &= I_r K_{ij}, \quad I_r = I_r^{\dagger}, \quad I_r^2 = 1, \\ K_{ij}x_j &= x_i K_{ij}, \quad K_{ij}x_k = x_k K_{ij}, \quad I_r x_i = -x_i I_r, \end{aligned} \tag{13}$$

for all $i \neq j \neq k \neq i$. The operators 1, K_{ij}, $K_{ijk} \equiv K_{ij}K_{jk}$, I_r, L_{ij}, and $L_{ijk} \equiv K_{ijk}I_r$, where i, j, k run over the set $\{1, 2, 3\}$, are the 12 elements of the dihedral group D_6.

By proceeding in a similar way in the present problem, we find the three differential-difference operators [Que96]

$$\begin{aligned} D_i &= \partial_i - \kappa a \sum_{j\neq i} \cot(ax_{ij}) K_{ij} \\ &\quad - \lambda a \left(\sum_{j \neq i} \cot(ay_{ij}) L_{ij} - \sum_{\substack{j,k \\ i\neq j\neq k\neq i}} \cot(ay_{jk}) L_{jk} \right), \end{aligned} \tag{14}$$

where $i = 1, 2, 3$. From their definition and Eq. (13), it is obvious that such operators are both anti-Hermitian and D_6-covariant, i.e., $D_i^{\dagger} = -D_i$, $K_{ij}D_j = D_i K_{ij}$, $K_{ij}D_k = D_k K_{ij}$, and $I_r D_i = -D_i I_r$, for all $i \neq j \neq k \neq i$, but that they do not commute among themselves. Their commutators are indeed given by

$$[D_i, D_j] = -a^2 \left(\kappa^2 + 3\lambda^2 - 4\kappa\lambda I_r\right) \sum_{k \neq i,j} \left(K_{ijk} - K_{ikj}\right), \qquad i \neq j, \tag{15}$$

and only vanish in the $a \to 0$ limit, i.e., for the CMW problem.

The operators D_i may be used to construct a generalized Hamiltonian with exchange terms

$$\begin{aligned} H_{exch} &\equiv -\sum_i \partial_i^2 + a^2 \sum_{\substack{i,j \\ i\neq j}} \csc^2(ax_{ij}) \kappa(\kappa - K_{ij}) \\ &\quad + 3a^2 \sum_{\substack{i,j \\ i \neq j}} \csc^2(ay_{ij}) \lambda(\lambda - L_{ij}) \\ &= -\sum_i D_i^2 + 6a^2 \left(\kappa^2 + 3\lambda^2\right) \\ &\quad + a^2 \left(\kappa^2 + 3\lambda^2 + 12\kappa\lambda I_r\right) \left(K_{123} + K_{132}\right). \end{aligned} \tag{16}$$

In those subspaces of Hilbert space wherein $(K_{ij}, L_{ij}) = (1, 1)$, $(1, -1)$, $(-1, 1)$, or $(-1, -1)$, the latter reduces to Hamiltonian (1) corresponding to $(g, f) = (\kappa(\kappa-1), \lambda(\lambda-1))$, $(\kappa(\kappa-1), \lambda(\lambda+1))$, $(\kappa(\kappa+1), \lambda(\lambda-1))$, or $(\kappa(\kappa+1), \lambda(\lambda+1))$, respectively.

From the operators D_i and the elements K_{ij}, L_{ij} of D_6, it is also possible to construct an alternative set of three Dunkl operators, i.e., three anti-Hermitian, commuting, albeit non-covariant, differential-difference operators

$$\begin{aligned}
\hat{D}_i &= D_i + i\kappa a \sum_{j\neq i} \alpha_{ij} K_{ij} + i\lambda a \left(\sum_{j\neq i} \beta_{ij} L_{ij} - \sum_{\substack{j,k \\ i\neq j\neq k\neq i}} \beta_{jk} L_{jk} \right), \\
& \alpha_{ij}, \beta_{ij} \in \mathbb{R},
\end{aligned} \tag{17}$$

in terms of which the generalized Hamiltonian with exchange terms, defined in Eq. (16), can be rewritten as $H_{exch} = -\sum_i \hat{D}_i^2$. By choosing for the α_{ij}'s the values previously considered for the Dunkl operators of the pure two-body problem [Ber93], namely $\alpha_{ij} = -\alpha_{ji} = -1$, $i < j$, one finds [Que96] that there are four equally acceptable choices for the remaining constants β_{ij}: $(\beta_{12}, \beta_{23}, \beta_{31}) = (-1, 1, 1)$, $(-1, 1, -1)$, $(-5/3, 1/3, 1/3)$, and $(-1/3, 5/3, -1/3)$.

The transformation properties under D_6 of the new operators $\hat{D}_i$ are given by

$$\begin{aligned}
K_{ij}\hat{D}_j - \hat{D}_i K_{ij} &= -i\kappa a \left(2\alpha_{ij} + \sum_{k\neq i,j} (\alpha_{ik} - \alpha_{jk}) K_{ijk} \right) \\
&\quad -i\lambda a \sum_{k\neq i,j} (\beta_{ik} - \beta_{jk}) I_r (K_{ijk} + 2K_{ikj}), \\
&\quad i \neq j, \\
\left[K_{ij}, \hat{D}_k \right] &= ia \left[\kappa (\alpha_{ik} - \alpha_{jk}) - \lambda (\beta_{ik} - \beta_{jk}) I_r \right] (K_{ijk} - K_{ikj}), \\
&\quad i \neq j \neq k \neq i, \\
\left\{ I_r, \hat{D}_i \right\} &= 2i\kappa a \sum_{j\neq i} \alpha_{ij} L_{ij} \\
&\quad + 2i\lambda a \left(\sum_{j\neq i} \beta_{ij} K_{ij} - \sum_{\substack{j,k \\ i\neq j\neq k\neq i}} \beta_{jk} K_{jk} \right).
\end{aligned} \tag{18}$$

The Hamiltonian with exchange terms H_{exch} can be related to a Hamiltonian $\mathcal{H}^{(\kappa,\lambda)}$ describing a one-dimensional system of three particles with $SU(n)$ "spins" (or colours in particle physics language), interacting via spin-dependent two and three-body potentials [Que96],

$$\mathcal{H}^{(\kappa,\lambda)} = -\sum_i \partial_i^2 + a^2 \sum_{\substack{i,j \\ i\neq j}} \csc^2(ax_{ij}) \kappa(\kappa - P_{ij})$$

$$+ 3a^2 \sum_{\substack{i,j \\ i\neq j}} \csc^2(ay_{ij})\lambda(\lambda - \tilde{P}_{ij}). \tag{19}$$

Here each particle is assumed to carry a spin with n possible values, and P_{ij}, $\tilde{P}_{ij} \equiv P_{ij}\tilde{P}$ are some operators acting only in spin space. The operator P_{ij} is defined as the operator permuting the ith and jth spins, while $\tilde{P}$ is a permutation-invariant and involutive operator, i.e., $\tilde{P}\sigma_i = \sigma_i^* \tilde{P}$, for some σ_i^* such that $P_{jk}\sigma_i^* = \sigma_i^* P_{jk}$ for all i, j, k, and $\sigma_i^{**} = \sigma_i$. For $SU(2)$ spins for instance, $\sigma_i = \pm 1/2$, $P_{ij} = (\sigma_i^a \sigma_j^a + 1)/2$, where σ^a, $a = 1$, 2, 3, denote the Pauli matrices, $\tilde{P}$ may be taken as 1 or $\sigma_1^1\sigma_2^1\sigma_3^1$, and accordingly $\sigma_i^* = \sigma_i$ or $-\sigma_i$. The operators P_{ij} and $\tilde{P}$ satisfy relations similar to those fulfilled by K_{ij} and I_r (cf. Eq. (13)), with x_i and $-x_i$ replaced by σ_i and σ_i^* respectively. Hence 1, P_{ij}, $P_{ijk} \equiv P_{ij}P_{jk}$, $\tilde{P}$, $\tilde{P}_{ij}$, and $\tilde{P}_{ijk} \equiv P_{ijk}\tilde{P}$ realize the dihedral group D_6 in spin space. Such a realization will be referred to as $D_6^{(s)}$ to distinguish it from the realization $D_6^{(c)}$ in coordinate space, corresponding to K_{ij} and I_r.

The Hamiltonian $\mathcal{H}^{(\kappa,\lambda)}$ remains invariant under the combined action of D_6 in coordinate and spin spaces (to be referred to as $D_6^{(cs)}$), since it commutes with both $K_{ij}P_{ij}$ and $I_r\tilde{P}$. Its eigenfunctions corresponding to a definite eigenvalue therefore belong to a (reducible or irreducible) representation of $D_6^{(cs)}$. For indistinguishable particles that are bosons (resp. fermions), only those irreducible representations of $D_6^{(cs)}$ that contain the symmetric (resp. antisymmetric) irreducible representation of the symmetric group S_3 should be considered. There are only two such inequivalent representations, which are both one-dimensional and denoted by A_1 and B_1 (resp. A_2 and B_2) [Ham62]. They differ in the eigenvalue of $I_r\tilde{P}$, which is equal to $+1$ or -1, respectively.

In such representations, for an appropriate choice of the parameters κ, λ, $\mathcal{H}^{(\kappa,\lambda)}$ can be obtained from H_{exch} by applying some projection operators. Let indeed $\Pi_{B\pm}$ (resp. $\Pi_{F\pm}$) be the projection operators that consist in replacing K_{ij} and I_r by P_{ij} (resp. $-P_{ij}$) and $\pm\tilde{P}$, respectively, when they are at the right-hand side of an expression. It is obvious that $\Pi_{B\pm}(H_{exch}) = \mathcal{H}^{(\kappa,\pm\lambda)}$, and $\Pi_{F\pm}(H_{exch}) = \mathcal{H}^{(-\kappa,\pm\lambda)}$. If H_{exch} has been diagonalized on a basis of functions depending upon coordinates and spins, then its eigenfunctions $\Psi(\boldsymbol{x},\boldsymbol{\sigma})$ are also eigenfunctions of $\mathcal{H}^{(\kappa,\pm\lambda)}$ (resp. $\mathcal{H}^{(-\kappa,\pm\lambda)}$) provided that $(K_{ij} - P_{ij})\Psi(\boldsymbol{x},\boldsymbol{\sigma}) = 0$ (resp. $(K_{ij} + P_{ij})\Psi(\boldsymbol{x},\boldsymbol{\sigma}) = 0$) and $(I_r \mp \tilde{P})\Psi(\boldsymbol{x},\boldsymbol{\sigma}) = 0$.

In conclusion, the three-body generalization of the Sutherland problem with internal degrees of freedom, corresponding to the Hamiltonian $\mathcal{H}^{(\kappa,\lambda)}$, is directly connected with the corresponding problem with exchange terms, governed by the Hamiltonian H_{exch}. The exchange operator formalism developed for the latter should therefore be relevant to a detailed study of the former.

References

[And84] A. A. Andrianov, N. V. Borisov and M. V. Ioffe, *Phys. Lett.* A **105** (1984) 19; A. A. Andrianov, N. V. Borisov, M. I. Eides and M. V. Ioffe, *Phys. Lett.* A **109** (1985) 143.

[Ber93] D. Bernard, M. Gaudin, F. D. M. Haldane and V. Pasquier, *J. Phys.* A **26** (1993) 5219.

[Bri92] L. Brink, T. H. Hansson and M. A. Vasiliev, *Phys. Lett.* B **286** (1992) 109.

[Cal69] F. Calogero, *J. Math. Phys.* **10** (1969) 2191, 2197; **12** (1971) 419.

[Cal74] F. Calogero and C. Marchioro, *J. Math. Phys.* **15** (1974) 1425.

[Dun89] C. F. Dunkl, *Trans. Am. Math. Soc.* **311** (1989) 167.

[Ham62] M. Hamermesh, *Group Theory* (Addison-Wesley, Reading, Mass., 1962).

[Ols83] M. A. Olshanetsky and A. M. Perelomov, *Phys. Rep.* **94** (1983) 313.

[Pol92] A. P. Polychronakos, *Phys. Rev. Lett.* **69** (1992) 109.

[Que95] C. Quesne, *Mod. Phys. Lett.* A **10** (1995) 1323.

[Que96] C. Quesne, *Europhys. Lett.* **35** (1996) 407.

[Que97] C. Quesne, *Phys. Rev.* A **55** (1997) 3931.

[Ruh95] W. Rühl and A. Turbiner, *Mod. Phys. Lett.* A **10** (1995) 2213.

[Sta89] R. P. Stanley, *Adv. Math.* **77** (1989) 76.

[Sut71] B. Sutherland, *Phys. Rev.* A **4** (1971) 2019; **5** (1972) 1372; *Phys. Rev. Lett.* **34** (1975) 1083.

[Wol74] J. Wolfes, *J. Math. Phys.* **15** (1974) 1420.